Chapter 3

First Order Ordinary Differential equation

Solve first order ordinary differential Equation

$$\frac{dy(x)}{dx} = f(x, y) \tag{3.1}$$

for function $y(x)$ with initial condition $y(x_0) = y_0$. This can be solved by recursion relation starting from y_0 at x_0.

Hamilton's equation of motion;

$$\begin{aligned} \frac{dq}{dt} &= \frac{\partial H(q,p,t)}{\partial q}, \\ \frac{dp}{dt} &= -\frac{\partial H(q,p,t)}{\partial q}. \end{aligned}$$

Newton's equation of motion;

$$\begin{aligned} m\frac{d^2x}{dt^2} &= f(t,x,v), \\ \frac{dx}{dt} &= v, \\ \frac{dv}{dt} &= f(t,x,v)/m. \end{aligned}$$

3.1 Euler's Method

Using two point differential form, the recursion relation of Euler's method is

$$y_{n+1} = y_n + hf(x_n, y_n) + O(h^2). \tag{3.2}$$

Since $y' = f(x, y(x))$, we have

$$y'' = \frac{d}{dx}f(x,y) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f. \tag{3.3}$$

Thus from Taylor expansion

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n + \frac{h^3}{3!}y'''_n + \cdots, \tag{3.4}$$

we can get higher order recursion relation

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2}\left[\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right]_n + O(h^3). \tag{3.5}$$

3.2 Using Extrapolation

Implicit methods using integral of f,

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x,y)dx. \tag{3.6}$$

Extrapolating $f(x, y)$ using two points of x_n and x_{n-1} and then integrating $f(x, y)$ for $x_n \le x \le x_{n+1}$, we get Adams-Bashforth 2-step Method

$$y_{n+1} = y_n + \frac{h}{2}(3f_n - f_{n-1}) + O(h^3). \tag{3.7}$$

Extension to multi-step method using multi-points extrapolation such as four points (x_n, x_{n-1}, x_{n-2}, x_{n-3}) cubic extrapolation and then integrating $f(x, y)$ for $x_n \le x \le x_{n+1}$:

$$y_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}) + O(h^4) \tag{3.8}$$

for 4-step method.

3.3 Using Interpolation

Adams-Moulton Method: Use interpolation of $f(x, y)$ including x_n and x_{n+1} points. Using three points of x_{n-1}, x_n, and x_{n+1}, the Adams-Moulton 2-step method is

$$y_{n+1} = \frac{h}{12}(5f_{n+1} + 8f_n - f_{n-1}) + O(h^4). \tag{3.9}$$

Using four points of x_{n-2}, x_{n-1}, x_n, and x_{n+1} it becomes 3-step method:

$$y_{n+1} = \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}) + O(h^5). \tag{3.10}$$

The unknown value of y_{n+1} is involved in Adams-Moulton method through $f_{n+1} = f(x_{n+1}, y_{n+1})$. For this problem use Predictor-Corrector Algorithm.

3.4 Runge-Kutta Methods

Use three points Simpson rule for the integral of $f(x, y)$ from y_n to y_{n+1} with three points interpolation of x_n, $x_{n+1/2}$, and x_{n+1}.

$$\begin{aligned} y_{n+1} &= y_n + \int_{y_n}^{y_{n+1}} f(x, y)dx \\ &\approx y_n + \frac{h}{6}\Big[f(x_n, y_n) + 4f(x_{n+1/2}, y_{n+1/2}) + f(x_{n+1}, y_{n+1})\Big] + O(h^5). \end{aligned} \tag{3.11}$$

This equation involves unknown values of $y_{n+1/2}$ and y_{n+1}.

Three point third order Runge-Kutta method;

$$\begin{aligned} k_1 &= hf(x_n, y_n), \\ k_2 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1), \\ k_3 &= hf(x_n + h, y_n - k_1 + 2k_2), \\ y_{n+1} &= y_n + \frac{1}{6}(k_1 + 4k_2 + k_3) + O(h^4). \end{aligned} \tag{3.12}$$

Three point fourth order Runge-Kutta method;

$$\begin{aligned} k_1 &= hf(x_n, y_n), \\ k_2 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1), \end{aligned}$$

$$
\begin{aligned}
k_3 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2), \\
k_4 &= hf(x_n + h, y_n + k_3), \\
y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k4) + O(h^5). \qquad (3.13)
\end{aligned}
$$

3.5 Stability

Since the error accumulates as integrate through from initial point to final point using the recurrence relation, the direction of integration may be vital for some function of $f(x, y)$. For example, $\frac{dy}{dx} = f(x, y) = -y$ has the solution $y(x) = e^{-x}$ which becomes smaller and smaller fast as x becomes larger. If we integrate out starting from $x = 0$ with $y(0) = 1$, then the accumulated error could be larger than the value of y_N at large x_N. If we integrate in from x_N with y_N, then the accumulated error could be much smaller than y_1 at small x_1.